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A note on the equality of algebraic and geometric D-brane charges in WZW models*

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Abstract: The algebraic definition of charges for symmetry-preserving D-branes in Wess-Zumino-Witten models is shown to coincide with the geometric definition, for all simple Lie groups. The charge group for such branes is computed from the ambiguities inherent in the geometric definition.

Keywords: Sigma Models, D-branes, Differential and Algebraic Geometry.

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Contents

1. Introduction

We consider symmetry-preserving D-branes in a Wess-Zumino-Witten model on a compact, connected, simply-connected simple Lie group G. Let T be a maximal torus, and let $\mathfrak{g}$ and $\mathfrak{t}$ be the respective Lie algebras. Classically, the symmetry-preserving D-branes of the WZW model on G at level k coincide, as submanifolds, with certain conjugacy classes. The quantised classes are in bijection with the integrable highest weight modules at level k of the associated affine Lie algebra $\hat{\mathfrak{g}}$.

Every conjugacy class intersects any chosen maximal torus in a finite number of points. If $h \in \mathrm{T}$ is such a point for a conjugacy class corresponding to a (symmetry-preserving) D-brane, then we may identify the brane manifold with G / Z, embedded in G, where Z is the centraliser of h [1]. When h is a regular element, $\mathrm{Z} = \mathrm{T}$, and we will call the brane regular. These are the symmetry-preserving branes of maximal dimension, and correspond to weights (via the exponential map) in the interior of the affine fundamental alcove[1] (at level k). When h is singular, $\mathrm{Z} \supset \mathrm{T}$, and we refer to these (lower dimensional) branes as singular. They correspond to weights on the boundary of the fundamental alcove.

There are at least two types of definition of D-brane charge in the literature. In [2], Fredenhagen and Schomerus introduced what we shall refer to as the algebraic definition, based on the conformal field theory approach "coupled" with renormalisation flow methods. If $\hat{\lambda}$ is the highest weight of the integrable highest weight module labelling a D-brane, then their definition amounts to the charge

$$Q_{\mathrm{alg}}(\lambda) = \dim L(\lambda), \tag{1.1}$$

where $L(\lambda)$ is the irreducible highest weight module of $\mathfrak{g}$ with (dominant integral) highest weight λ, the projection of $\hat{\lambda}$ onto the weight space $\mathfrak{t}^*$ of the horizontal subalgebra. To be

[1] To be precise, we mean the simplex in $\mathfrak{t}^*$ corresponding to the affine fundamental alcove.

precise, their charge is this dimension modulo some integer which needs to be determined from the fusion rules. This integer was determined in [2, 3] for symmetry-preserving branes in (supersymmetric) WZW models on $\mathrm{SU}(n)$, and for the other simple Lie groups in [4]. For instance, such branes on $\mathrm{SU}(2)$ admit charges valued modulo $k+2$. This charge definition appears to fit in nicely with the notion that D-brane charges should be classified by twisted K-theory, in that the charge groups are in agreement with the relevant K-theoretic results which are presently known [3, 5, 6].

By contrast, the geometric definition is a little more involved, and we shall review its construction, and a little of its history, in section 2. We take some care with this construction, in particular with the "U (1) flux", so as to isolate the ambiguities involved in defining the geometric brane charge. In section 3, we compute geometric charges for the regular branes in $\mathrm{SU}(3)$ in order to provide a non-trivial example (that is, not $\mathrm{SU}(2)$) showing that the geometric charge coincides with the algebraic charge. This obviously suggests that the algebraic and geometric charges might coincide for all (simple) groups. Section 4 is devoted to proving this. In section 5, we return to the ambiguities in the geometric charge definition and investigate what this implies for D-brane charge groups.

2. Geometric charges

WZW models come equipped with an integral 3-form $H \in \mathrm{H}^3(\mathrm{G};\mathbb{Z}) \cong \mathbb{Z}$ of period k, and each D-brane (labelled by λ) comes with a 2-form $\omega_\lambda \in \Omega^2(\mathrm{G}/\mathrm{Z})$ whose derivative is H restricted to the D-brane [7]. Locally then $H = \mathrm{d}B$, so $F_\lambda = B - \omega_\lambda$ represents a degree-two cohomology class wherever it is defined. F_λ is sometimes referred to as the U (1) flux [9]. The coupling of the Ramond-Ramond fields in string theory [8, 9] then suggests a charge

$$Q'_{\mathrm{geo}}(\lambda) = \int_{\mathrm{G}/\mathrm{Z}} e^{F_\lambda}, \tag{2.1}$$

where the integration is over the entire brane G / Z. This definition has been known for some time to require modification, and a careful analysis of certain quantum anomalies suggests the correct modification [10]. We will consider the modified charge shortly (eq. (2.3)), but first we would like to examine the 2-form F_λ in more detail.

For the case $\mathrm{G} = \mathrm{SU}(2)$ whose D-brane charges have been computed many times [9, 11, 12], explicit calculations are elementary. However, in general, there is an issue regarding how one defines an appropriate B on the entire brane. H is exact there, but in the absence of further constraints on B, adding a closed form to any candidate B would allow us to produce any numerical value we like for the geometric charge.

Let us instead follow [11, 13] and note in that the standard procedure leading to brane quantisation, integers associated with each (allowed) brane naturally arise. These are the possible ambiguities in the action, and take the form

$$Q_{\mathrm{amb}}(\lambda, S^2, M) = \int_M H - \int_{S^2} \omega_\lambda\,, \tag{2.2}$$

where S^2 is an arbitrary 2-sphere in the brane and M is an arbitrary 3-chain whose boundary is S^2.

If one considers SU (2), whose group manifold is a 3-sphere, $S^2 = \mathrm{SU}(2) / \mathrm{U}(1)$ to be a regular brane,[2] and M to be a side of the 3-sphere bounded by the brane, Q_{amb} can be easily computed and is found to agree remarkably with the algebraic charge (to be precise, we find $Q_{alg} - Q_{\text{amb}} = 1$). Moreover, if we take M to be the other side of the brane, then Q_{amb} changes by k, the period of H over the 3-sphere. With the usual quantum shift, $k \to k + \mathrm{h}^\vee$ ($\mathrm{h}^\vee$ the dual Coxeter number of $\mathfrak{g}$), we conclude that Q_{amb} is therefore defined modulo $k+2$, again in remarkable agreement with the algebraic result. It appears then that for SU (2), Q_{amb} defines a geometric charge for branes.

Eq. (2.1) suggests that to generalise this to other groups, we need to trivialise H, and exponentiate. We shall do exactly this to define F_λ, and thus $Q'_{\text{geo}}(\lambda)$. Consider therefore the brane G / Z. When Z is a centraliser of a torus,[3] the homology of G / Z has no torsion and vanishes in odd degrees [15]. Choose a set $\{S_i^2\}$ of generators of $\mathrm{H}_2(\mathrm{G} / \mathrm{Z}; \mathbb{Z})$ (we may take 2-spheres as generators by the Hurewicz isomorphism), and let M_i be a 3-chain whose boundary is S_i^2 (since $\mathrm{H}_2(G;\mathbb{Z}) = 0$, we may choose them to be 3-cells). As the brane generally has non-vanishing second homology group, the M_i are not usually contained within the brane. Let $\mathcal{C}$ be the complex obtained by attaching each 3-cell M_i to the brane along each corresponding S_i^2. Then $\mathrm{H}_2(\mathcal{C};\mathbb{Z})$ clearly vanishes, and the Mayer-Vietoris sequence for attaching [16] shows that $\mathrm{H}_3(\mathcal{C};\mathbb{Z})$ vanishes also. As attaching 3-cells leaves the first homology group invariant, it follows from the universal coefficient theorem that $\mathcal{C}$ has vanishing cohomology in degrees 2 and 3. Thus B may be defined on $\mathcal{C}$, so $F_\lambda = B - \omega_\lambda$ is defined on the whole brane. Note we are free to add any closed 2-form on $\mathcal{C}$ to B, but these are all exact on $\mathcal{C}$, hence exact on the brane. Thus F_λ is well-defined in cohomology, so $Q'_{\text{geo}}(\lambda)$ is also well-defined.[4]

Note that $Q_{\text{amb}}(\lambda, S_i^2, M_i)$ is just the period of F_λ over S_i^2. A little Morse theory suggests [17] that for the S_i^2 we may take (translates of) certain conjugacy classes of $\mathrm{SU}(2)_{\alpha_i}$, the embedded SU (2)-subgroups corresponding to the simple roots α_i. The standard choice is now to take M_i to be one side of (the translated) $\mathrm{SU}(2)_{\alpha_i}$, thus reducing the computation of Q_{amb} for general simple groups to the SU (2) case. This is a standard computation and gives

$$Q_{\text{amb}}(\lambda, S_i^2, M_i) = \langle \lambda, \alpha_i^\vee \rangle .$$

That is, the period of F_λ over the chosen homology generator S_i^2 is just the Dynkin label λ_i.

As noted above, the algebraic and geometric charges are in close agreement for SU (2). The computation for SU (3) does not seem to appear in the literature. It may be performed without fuss (for regular branes) using Schubert calculus, and we shall outline it in section 3. We find that this time, the algebraic and (naïve) geometric charge differ by a non-constant amount, suggesting that the geometric charge needs modifying.

[2]The argument extends trivially to singular branes by taking S^2 to be a degenerate 2-sphere of zero radius.

[3]This is almost always true, as follows from a theorem of Steinberg [14]. In fact, at each level, there are at most $r+1$ exceptions, where r is the rank of G. We are chiefly concerned with regular branes, and so we shall not consider these exceptions any further.

[4]Given $\mathcal{C}$ — see section 5.

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As noted above, such a modification was suggested in [10], based on the cancellation of quantum anomalies. There it was also shown that their modified charge has a natural K-theoretic interpretation. Their result, after specialising to the case we are interested in, amounts to

$$Q_{\text{geo}}(\lambda) = \int_{\mathrm{G}/\mathrm{Z}} e^{F_\lambda}\, \mathrm{Td}\left(T\left(\mathrm{G}/\mathrm{Z}\right)\right), \tag{2.3}$$

where $T(M)$ denotes the tangent bundle of the manifold M, and $\mathrm{Td}(E)$ denotes the Todd class of the vector bundle E.

In [12] this charge is computed for $\mathrm{SU}(2)$ and is found to exactly agree with the algebraic charge (no discrepancy of 1). In section 3 we will extend this to regular branes in $\mathrm{SU}(3)$, using Schubert calculus (although we need the Chern classes of the tangent bundle of the brane manifold), and find again exact agreement. We therefore adopt this definition as our geometric brane charge. In section 4, we will prove that the algebraic and geometric charges coincide for all simple Lie groups (but without invoking any Schubert theory).

3. $\mathrm{SU}(3)$ calculations

We now digress to perform a couple of computations regarding regular brane charges in $\mathrm{SU}(3)$ using the technology of Schubert calculus [18]. This approach is convenient as the brane manifolds are examples of (complete) flag manifolds. The independent homology cycles of the regular branes $\mathrm{SU}(3)/\mathrm{U}(1)^2$, called Schubert cells, are labelled by the elements of the Weyl group, $\mathrm{W} = \mathrm{S}_3$, of $\mathrm{SU}(3)$. The 2-spheres S_i^2 associated with the simple roots α_i (considered in section 2), in fact correspond to the simple reflections w_i.

The computational utility of Schubert calculus becomes apparent when we take Poincaré duals and consider the cohomology ring. The Schubert cell $X_w \in \mathrm{H}_{\ell(w)}\left(\mathrm{SU}(3)/\mathrm{U}(1)^2;\mathbb{Z}\right)$ corresponding to $w \in \mathrm{S}_3$ ($\ell(w)$ is the length of w) has Poincaré dual denoted by $p_{w_0 w} \in \mathrm{H}^{\ell(w_0 w)}\left(\mathrm{SU}(3)/\mathrm{U}(1)^2;\mathbb{Z}\right)$, where $w_0 = w_1 w_2 w_1 = w_2 w_1 w_2$ is the longest element of S_3. In turn, the p_w may be expressed as polynomials (called Schubert polynomials) in certain generators $x_1, x_2 \in \mathrm{H}^2\left(\mathrm{SU}(3)/\mathrm{U}(1)^2;\mathbb{Z}\right)$. These generators are chosen to be naturally permuted by S_3. To be specific, we have in the usual representation,

$$\begin{aligned} p_e &= 1 & p_{w_1} &= x_1 & p_{w_2} &= x_1 + x_2 \\ p_{w_1 w_2} &= x_1 x_2 & p_{w_2 w_1} &= x_1^2 & p_{w_1 w_2 w_1} &= x_1^2 x_2 . \end{aligned}$$

Finally, we also need the cohomology ring of $\mathrm{SU}(3)/\mathrm{U}(1)^2$ which is a special case of a famous result of Borel:

$$\mathrm{H}^*\left(\mathrm{SU}(3)/\mathrm{U}(1)^2;\mathbb{Z}\right) \cong \frac{\mathbb{Z}[x_1, x_2]}{\left\langle x_1^2 + x_1 x_2 + x_2^2, x_1^3 \right\rangle}.$$

The general result (Theorem 1) will be important in section 4. Note that the p_w given above generate the integral cohomology of $\mathrm{SU}(3)/\mathrm{U}(1)^2$.

We will compute both $Q'_{\text{geo}}(\lambda)$ and $Q_{\text{geo}}(\lambda)$ for regular branes in $\mathrm{SU}(3)$. First, we determine F_λ in terms of the Schubert basis. We have

$$\lambda_1 = \int_{X_{w_1}} F_\lambda = \int_{\mathrm{SU}(3)/\mathrm{U}(1)^2} F_\lambda \wedge p_{w_1 w_2} = \int_{\mathrm{SU}(3)/\mathrm{U}(1)^2} F_\lambda \wedge x_1 x_2$$

and similarly, $\lambda_2 = \int_{\mathrm{SU}(3)/\,\mathrm{U}(1)^2} F_\lambda \wedge x_1^2$. A little algebra now gives

$$F_\lambda = (\lambda_1 + \lambda_2)\, x_1 + \lambda_2 x_2 \,.$$

Hence, making much use of the form of the cohomology ring,

$$Q'_{\mathrm{geo}}(\lambda) = \frac{1}{3!} \int_{\mathrm{SU}(3)/\,\mathrm{U}(1)^2} F_\lambda^3 = \frac{1}{2}\left(\lambda_1^2 \lambda_2 + \lambda_1 \lambda_2^2\right) \,, \tag{3.1}$$

which is not the dimension of the irreducible representation of highest weight λ.

We therefore turn to the computation of $Q_{\mathrm{geo}}(\lambda)$. For E a complex vector bundle, the homogeneous components, $\mathrm{Td}_n(E)$, of the Todd class $\mathrm{Td}(E)$ can be expressed as polynomials (the Todd polynomials) in the Chern classes, $\mathrm{c}_i(E)$, of E. The first few are [19]:

$$\begin{aligned}
\mathrm{Td}_0(E) &= 1 & \mathrm{Td}_1(E) &= \frac{1}{2}\mathrm{c}_1(E) \\
\mathrm{Td}_2(E) &= \frac{1}{12}\left[\mathrm{c}_1(E)^2 + \mathrm{c}_2(E)\right] & \mathrm{Td}_3(E) &= \frac{1}{24}\mathrm{c}_1(E)\,\mathrm{c}_2(E) \,.
\end{aligned}$$

The contributing part of the integrand of $Q_{\mathrm{geo}}(\lambda)$ is therefore

$$\begin{aligned}
&\mathrm{Td}_3(E) + \mathrm{Td}_2(E)\, F_\lambda + \frac{1}{2}\mathrm{Td}_1(E)\, F_\lambda^2 + \frac{1}{3!}\mathrm{Td}_0(E)\, F_\lambda^3 = \\
&\quad = \frac{1}{24}\mathrm{c}_1(E)\,\mathrm{c}_2(E) + \frac{1}{12}\left[\mathrm{c}_1(E)^2 + \mathrm{c}_2(E)\right] F_\lambda + \frac{1}{4}\mathrm{c}_1(E)\, F_\lambda^2 + \frac{1}{6} F_\lambda^3 \,,
\end{aligned}$$

where $E = T\left(\mathrm{SU}(3)\,/\,\mathrm{U}(1)^2\right)$, considered as a complex vector bundle.

Evidently we need the Chern classes of this tangent bundle. These may be computed using a result of Borel and Hirzebruch which we shall use in section 4 (Theorem 2). We leave it as an easy exercise to show from this result that[5]

$$\mathrm{c}_1\left(T\left(\mathrm{SU}(3)/\mathrm{U}(1)^2\right)\right) = 4x_1 + 2x_2$$

and

$$\mathrm{c}_2\left(T\left(\mathrm{SU}(3)/\mathrm{U}(1)^2\right)\right) = 6x_1^2 + 6x_1 x_2 \,.$$

It immediately follows that

$$\begin{aligned}
\frac{1}{24}\mathrm{c}_1(E)\,\mathrm{c}_2(E) &= x_1^2 x_2 \,, \\
\frac{1}{12}\left[\mathrm{c}_1(E)^2 + \mathrm{c}_2(E)\right] F_\lambda &= \frac{3}{2}(\lambda_1 + \lambda_2)\, x_1^2 x_2 \,, \\
\frac{1}{4}\mathrm{c}_1(E)\, F_\lambda^2 &= \frac{1}{2}\left(\lambda_1^2 + 4\lambda_1\lambda_2 + \lambda_2^2\right) x_1^2 x_2 \,,
\end{aligned}$$

and from eq. (3.1) that

$$\frac{1}{6} F_\lambda^3 = \frac{1}{2}\left(\lambda_1^2 \lambda_2 + \lambda_1 \lambda_2^2\right) x_1^2 x_2 \,.$$

[5]Note that $\mathrm{c}_3\left(T\left(\mathrm{SU}(3)\,/\,\mathrm{U}(1)^2\right)\right) = 6x_1^2 x_2$ is elementary, but is not required for this calculation.

Thus, the geometric charge is

$$\begin{aligned} Q_{\mathrm{geo}}(\lambda) &= \int_{\mathrm{SU}(3)/\,\mathrm{U}(1)^2} \left[1+\frac{3}{2}(\lambda_1+\lambda_2)+\frac{1}{2}\left(\lambda_1^2+4\lambda_1\lambda_2+\lambda_2^2\right)+\frac{1}{2}\left(\lambda_1^2\lambda_2+\lambda_1\lambda_2^2\right)\right] x_1^2 x_2 \\ &= \frac{1}{2}(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2) = \dim L(\lambda)\,. \end{aligned}$$

So the algebraic and geometric charges agree in this case.

4. Proof: the charges coincide generally

We now prove that the charges $Q_{\mathrm{alg}}(\lambda)$ and $Q_{\mathrm{geo}}(\lambda)$ in fact coincide for symmetry-preserving WZW D-branes in compact, connected, simply-connected, simple Lie groups. Actually, for the purposes of clarity, we will only prove this for regular branes. The computation itself is also essentially contained in [20], though there the motivation is of course purely mathematical. The extension to singular branes is not too demanding, and is left as an exercise for the interested reader.

The proof is based on two results which were alluded to in section 3, one due to Borel [21], and the other to Borel and Hirzebruch [22]. Before presenting these results, it is convenient to introduce a formalism in which brane cohomology is interpreted in terms of Lie-theoretic data. We have a natural sequence of isomorphisms,

$$\mathrm{H}_2(\mathrm{G}\,/\,\mathrm{T};\mathbb{Z}) \cong \pi_1(\mathrm{T}) \cong \ker\{\exp : \mathfrak{t} \to \mathrm{T}\} = \mathrm{Q}^\vee\,,$$

where the first is transgression composed with the isomorphism between $\mathrm{H}_1(\mathrm{T};\mathbb{Z})$ and $\pi_1(\mathrm{T})$, or alternatively, Hurewicz composed with the isomorphism from the homotopy long exact sequence, and $\mathrm{Q}^\vee$ is the coroot (integral) lattice of $\mathfrak{g}$. It follows now that in a very natural sense,

$$\mathrm{H}^2(\mathrm{G}\,/\,\mathrm{T};\mathbb{Z}) = \mathrm{Hom}\left(\mathrm{H}_2(\mathrm{G}\,/\,\mathrm{T};\mathbb{Z}),\mathbb{Z}\right) \cong \left(\mathrm{Q}^\vee\right)^* = \mathrm{P}\,,$$

where P is the weight lattice of $\mathfrak{g}$.

Thus we may say that the fundamental weights generate the second cohomology group of the regular branes. With this formalism, Borel's famous result on the cohomology of G / T is:

Theorem 1 ([21]) *Let Λ_i denote the fundamental weights of* G *(simple of rank r), and* W *denote the Weyl group of* G*. Then the rational cohomology ring of* G / T *is generated by the Λ_i (and the unit) modulo the* W*-invariant polynomials of positive degree. Specifically,*

$$\mathrm{H}^*(\mathrm{G}\,/\,\mathrm{T};\mathbb{Q}) = \frac{\mathbb{Q}[\Lambda_1,\ldots,\Lambda_r]}{\mathcal{I}_+}\,,$$

where $\mathcal{I}_+$ is the ideal generated by the W*-invariants of positive degree.*

When the cohomology of G *has no torsion, that is* $\mathrm{G} = \mathrm{SU}(r+1)$ *or* $\mathrm{Sp}(2r)$*, then the result also holds over the integers.*

The other result concerns the characteristic classes of the tangent bundle to G/T. Via the splitting principle [16], this bundle (treated as a complex vector bundle) is cohomologically equivalent to a direct sum of line bundles. The Chern classes of the bundle are then the elementary symmetric polynomials in the first Chern classes of the associated line bundles.

Theorem 2 ([22]) *With the above formalism, the first Chern classes of the line bundles associated to $T(\mathrm{G}/\mathrm{T})$ under the splitting principle, are the positive roots of* G. *Thus if Δ_+ denotes the set of positive roots, the total Chern class of $T(\mathrm{G}/\mathrm{T})$ is given by*

$$\mathrm{c}\left(T\left(\mathrm{G}/\mathrm{T}\right)\right) = \prod_{\alpha\in\Delta_+}(1+\alpha)\,.$$

We can now show that the algebraic and geometric charges coincide. First however, note the beautiful interpretation of the cohomology class of our 2-form F_λ in this formalism. As the periods of this form are just the Dynkin labels and the fundamental weights are generators of $\mathrm{H}^2(\mathrm{G}/\mathrm{T};\mathbb{Z})$, it follows that in this formalism, F_λ is identified with λ.

Consider the integrand of the geometric charge. We have [19]

$$\begin{aligned} e^\lambda \mathrm{Td}\left(T\left(\mathrm{G}/\mathrm{T}\right)\right) &= e^\lambda \prod_{\alpha\in\Delta_+}\frac{\alpha}{1-e^{-\alpha}} = e^\lambda \frac{\prod_{\alpha\in\Delta_+}e^{\alpha/2}}{\prod_{\alpha\in\Delta_+}\left(e^{\alpha/2}-e^{-\alpha/2}\right)}\prod_{\alpha\in\Delta_+}\alpha \\ &= \frac{e^{\lambda+\rho}}{\prod_{\alpha\in\Delta_+}\left(e^{\alpha/2}-e^{-\alpha/2}\right)}\prod_{\alpha\in\Delta_+}\alpha\,, \end{aligned}$$

where $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$ is the Weyl vector of G.

Note that $\prod_{\alpha\in\Delta_+}\alpha$ is cohomologically non-trivial, for it is the top Chern class:

$$\begin{aligned} \int_{\mathrm{G}/\mathrm{T}}\prod_{\alpha\in\Delta_+}\alpha &= \int_{\mathrm{G}/\mathrm{T}}\mathrm{c}_m\left(T\left(\mathrm{G}/\mathrm{T}\right)\right) = \int_{\mathrm{G}/\mathrm{T}}\mathrm{e}\left(T\left(\mathrm{G}/\mathrm{T}\right)\right) \\ &= \chi\left(\mathrm{G}/\mathrm{T}\right) = \left|\mathrm{W}\right|, \end{aligned}$$

the last equality from the fact that the odd Betti numbers vanish and the homology classes are in correspondence with W [17]. Here $m = |\Delta_+|$ (so $\dim \mathrm{G} = 2m+r$), $\mathrm{e}(E)$ is the Euler class of E and $\chi(M)$ is the Euler characteristic of M.

Now the only component of the integrand which contributes is that in $\mathrm{H}^{2m}(\mathrm{G}/\mathrm{T};\mathbb{R})$. Since $\prod_{\alpha\in\Delta_+}\alpha$ generates this cohomology group, we should only have to consider the degree-zero term in the prefactor

$$\frac{e^\lambda}{\prod_{\alpha\in\Delta_+}\left(1-e^{-\alpha}\right)}\,.$$

This prefactor, as a function on $\mathfrak{t}$, has a pole of order m at the origin, and so it is not straight-forward to extract its degree-zero term. However, we recognise this as the character of the Verma module for $\mathfrak{g}$ of highest weight λ. This is an infinite-dimensional module, hence the non-analytic behaviour. Finite-dimensional modules, on the other hand, have characters which are analytic everywhere.

With this in mind, let us note that the product of the roots is anti-invariant under W:

$$w\left(\prod_{\alpha\in\Delta_+}\alpha\right) = (-1)^{\ell(w)}\prod_{\alpha\in\Delta_+}\alpha = \det w\prod_{\alpha\in\Delta_+}\alpha\,.$$

As G / T is W-invariant (up to a change in orientation), we have

$$\begin{aligned}\int_{\mathrm{G}\,/\,\mathrm{T}} e^{\lambda}\prod_{\alpha\in\Delta_+}\frac{\alpha}{1-e^{-\alpha}} &= \det w\int_{\mathrm{G}\,/\,\mathrm{T}} w\left(e^{\lambda+\rho}\prod_{\alpha\in\Delta_+}\frac{\alpha}{e^{\alpha/2}-e^{-\alpha/2}}\right)\\ &= \det w\int_{\mathrm{G}\,/\,\mathrm{T}} e^{w(\lambda+\rho)-\rho}\prod_{\alpha\in\Delta_+}\frac{\alpha}{1-e^{-\alpha}}.\end{aligned}$$

We may therefore write

$$\begin{aligned}\int_{\mathrm{G}\,/\,\mathrm{T}} e^{\lambda}\,\mathrm{Td}\left(T\left(\mathrm{G}\,/\,\mathrm{T}\right)\right) &= \frac{1}{|\mathrm{W}|}\int_{\mathrm{G}\,/\,\mathrm{T}}\sum_{w\in\mathrm{W}}\det w\; e^{w(\lambda+\rho)-\rho}\prod_{\alpha\in\Delta_+}\frac{\alpha}{1-e^{-\alpha}}\\ &= \frac{1}{|\mathrm{W}|}\int_{\mathrm{G}\,/\,\mathrm{T}}\chi_\lambda\prod_{\alpha\in\Delta_+}\alpha\,,\end{aligned}$$

where we recognise Weyl's character formula for the character, χ_λ, of the irreducible module $L\left(\lambda\right)$ with highest weight λ. As we now have a finite-dimensional module, extracting the degree-zero term is trivial. Obviously,

$$\begin{aligned}Q_{\mathrm{geo}}\left(\lambda\right) &= \frac{1}{|\mathrm{W}|}\int_{\mathrm{G}\,/\,\mathrm{T}}\chi_\lambda\prod_{\alpha\in\Delta_+}\alpha = \frac{\dim L\left(\lambda\right)}{|\mathrm{W}|}\int_{\mathrm{G}\,/\,\mathrm{T}}\prod_{\alpha\in\Delta_+}\alpha\\ &= \dim L\left(\lambda\right) = Q_{\mathrm{alg}}\left(\lambda\right)\,,\end{aligned}$$

which completes our proof.

It is interesting to observe that in the integrand,

$$e^{\lambda+\rho}\prod_{\alpha\in\Delta_+}\frac{\alpha}{e^{\alpha/2}-e^{-\alpha/2}},$$

the product is W-invariant, and so by theorem 1, it is cohomologically equivalent to its zero-degree term. Therefore, we may replace the product by a constant, which is easily seen to be in fact 1. It follows that for our regular branes, the geometric charge in fact reduces to

$$Q_{\mathrm{geo}}\left(\lambda\right) = \int_{\mathrm{G}\,/\,\mathrm{T}} e^{\lambda+\rho}\,.$$

That is, the naïve geometric charge is only altered by the quantum shift $\lambda\to\lambda+\rho$. Therefore, we have the satisfying conclusion that the modified geometric charge of [10], in this special case, reduces to the "naïve" charge of [8] *after quantisation is taken into account.* One can of course check that the computation in section 3 is consistent with this observation.

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5. Charge group constraints

Let us briefly review the constraints on the charge group derived from the algebraic theory. These charges were considered with regard to brane condensation processes in [2]. There, a formal analogy with the Kondo model of condensed matter physics was exploited to suggest charge conservation conditions of the form

$$Q_{\text{alg}}(\lambda)\, Q_{\text{alg}}(\mu) = \sum_{\nu} \mathcal{N}_{\lambda\mu}{}^{\nu}\, Q_{\text{alg}}(\nu) \ , \tag{5.1}$$

where $\mathcal{N}_{\lambda\mu}{}^{\nu}$ denotes the fusion coefficients of the WZW theory (we assume symmetry-preserving branes for simplicity). These conditions are not usually satisfied over the integers, and instead are interpreted as holding in the charge group of the symmetry-preserving branes, which is of the form $\mathbb{Z}_x$. Under the assumption that these constraints are exhaustive, the integer x was thereby determined for $\mathrm{SU}(n)$ in [2, 3], and general simple Lie groups G in [4] as

$$x = \frac{k + \mathrm{h}^{\vee}}{\gcd\left\{k + \mathrm{h}^{\vee}, y\right\}} , \tag{5.2}$$

where $y \in \mathbb{Z}$ depends only on G and is explicitly given in table 1 of [4] (recall k is the level and $\mathrm{h}^{\vee}$ the dual Coxeter number of $\mathfrak{g}$). Subsequently [5], the computation of the relevant K-theories was reduced to a point where it could be directly compared with the formalism of [4], showing why the two approaches should agree. This relied heavily on the theorem of Freed, Hopkins and Teleman [23, 6] which says that twisted equivariant K-theory is in fact the fusion algebra.

Consider now the geometric definition of brane charge. We have labelled our brane by a weight λ which lies in the affine fundamental Weyl alcove, and shown that the charge is given by $\dim L(\lambda)$. This labelling arises [24] from the fact that symmetry-preserving branes correspond to conjugacy classes through some $h \in \mathrm{T}$, where (after quantisation) $h = \exp\{2\pi i y\}$, and $y \in \mathfrak{t}$ is given as the image of $(\lambda+\rho)/(k+\mathrm{h}^{\vee})$ under the canonical isomorphism $\mathfrak{t} \cong \mathfrak{t}^*$. Insisting that λ belong to the affine fundamental alcove fixes it uniquely, which is why we use it to label our brane, but it is clear that there are choices being made here. Indeed, a cursory examination of sections 2 and 4 should convince the reader that there is no reason why λ must be taken in the affine fundamental alcove.

First, $h \in \mathrm{T}$, and therefore $y \in \mathfrak{t}$, is only determined up to the action of W. Hence $\lambda \in \mathfrak{t}^*$ is ambiguous up to the shifted action of W, $w \cdot \lambda = w(\lambda+\rho) - \rho$. It follows that for the geometric charge to be well-defined, we must have[6]

$$\dim L(\lambda) = \det w \ \dim L(w \cdot \lambda) \tag{5.3}$$

for each integral λ in the fundamental alcove, and each $w \in \mathrm{W}$. The $\det w$ factor arises as w may reverse the orientation of the brane manifold. It is an easy exercise to prove that eq. (5.3) is always satisfied (over $\mathbb{Z}$).

[6] Of course we must extend dim, in the obvious way, as a polynomial form on $\mathfrak{t}^*$. The expression $L(\lambda)$ is not to be taken literally if λ is outside the fundamental chamber.

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Next, $\exp : \mathfrak{t} \to \mathrm{T}$ is not injective, hence $h \in \mathrm{T}$ only determines $y \in \mathfrak{t}$ up to the integral lattice, which is the coroot lattice $\mathrm{Q}^\vee$ (since G is simply-connected). Thus λ is also ambiguous up to translations by $(k + \mathrm{h}^\vee)$ times the coroot lattice. Together with the first ambiguity, we therefore find that λ is only determined up to the shifted action of the *affine* Weyl group (at level k), $\widehat{\mathrm{W}}_k$. Hence for the geometric charge to be well-defined,

$$\dim L(\lambda) = \det \widehat{w} \, \dim L(\widehat{w} \cdot \lambda) \ , \tag{5.4}$$

for all integral λ in the fundamental alcove (and hence all integral λ), and each $\widehat{w} \in \widehat{\mathrm{W}}_k$. These relations do not hold in general over $\mathbb{Z}$ and should be interpreted as constraints on the charge group $\mathbb{Z}_x$.

Recall that the affine Weyl group is used to compute the fusion coefficients via the Kac-Walton formula [25, 26]

$$\mathcal{N}_{\lambda\mu}{}^{\nu} = \sum_{\substack{\widehat{w} \in \widehat{\mathrm{W}}_k \\ \widehat{w} \cdot \nu \in \mathrm{P}_+}} \det \widehat{w} \, N_{\lambda\mu}{}^{\widehat{w} \cdot \nu} \ ,$$

where $N_{\lambda\mu}{}^{\nu}$ are the tensor-product multiplicities ($L(\lambda) \otimes L(\mu) \cong \oplus_\nu N_{\lambda\mu}{}^{\nu} L(\nu)$) and P_+ is the set of dominant integral weights. It therefore seems reasonable to compare the charge groups obtained from the constraints (5.4) with those of (5.1).

For $\mathrm{G} = \mathrm{SU}(2)$ at level k, (5.4) admits the charge group[7] $\mathbb{Z}_{2(k+2)}$, whereas (5.1) gives $\mathbb{Z}_{k+2}$. Numerical computations for the other groups (low ranks and levels) suggest that in fact the constraints (5.4) reproduce exactly the charge groups computed in [4], except for $\mathrm{Sp}(2n)$, n a power of 2 (note that $\mathrm{SU}(2) = \mathrm{Sp}(2)$). As one would expect, the exceptions noted there are not reproduced[8]. When these charge groups disagree, (5.4) gives $\mathbb{Z}_x$ where x is twice what eq. (5.2) predicts. This appears to be due to fixed points of the $\widehat{\mathrm{W}}_k$-action. For instance, a weight λ on the boundary of the (shifted) affine alcove is fixed by some reflection $\widehat{w} \in \widehat{\mathrm{W}}_k$. This leads to a constraint

$$2 \dim L(\lambda) = 0 \mod x \ ,$$

allowing this dimension to be non-zero in $\mathbb{Z}_x$ if x is even, whereas in the fusion ring, boundary weights vanish.

There is still another ambiguity we have not accounted for yet. Recall in section 2 that we constructed a complex $\mathcal{C}$, for regular branes, by attaching 3-cells to G / T along a basis of homology 2-spheres. The 3-cells correspond to "halves" of translated $\mathrm{SU}(2)$-subgroups of G, and are essentially chosen so that the periods of F_λ could be easily computed. However, we may attach any 3-chains, with the correct boundaries, that we like (in particular we could choose either half of the $\mathrm{SU}(2)$-subgroups). Whilst this will change $\mathcal{C}$, the periods of

[7]We have again accounted for the quantum shift.

[8]In light of comments in [5, 27], we would like to point out that at low level, the constraints (5.1), by themselves, give rise to parameters x which may differ from (5.2), and that these exceptions were *explicitly* noted in [4] to imply charge groups that are absurd from a K-theoretic viewpoint. Implicitly then, the constraints (5.1), as interpreted there, are not completely exhaustive at sufficiently low levels.

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F_λ can only change by a multiple of $(k+\mathrm{h}^\vee)$. As F_λ is identified with λ in the formalism of section 4, we conclude that for the purposes of $Q_{\mathrm{geo}}(\lambda)$, λ is ambiguous up to translations by $(k+\mathrm{h}^\vee)$ times the *weight* lattice P. That is, for the geometric charge to be well-defined, we must have

$$\dim L(\lambda) = \dim L\left(\lambda + (k+\mathrm{h}^\vee)\mu\right) , \tag{5.5}$$

for all $\lambda, \mu \in P$. Note that as $\mathrm{Q}^\vee \subseteq P$, the constraints (5.5) are stronger than (5.4).

For SU(2), we recover from these stronger constraints the correct charge group $\mathbb{Z}_{k+2}$. In fact, this ambiguity is what was used in [12, 13] to compute this charge group. Numerical computations for the other groups indicate that these constraints (5.5) also reproduce exactly the charge groups of [4], except for Sp$(2n)$, this time for n *not* a power of 2 (now x may be half of what (5.2) predicts). These weight lattice charge groups were also investigated in [4] in the context of symmetries of the charges obtained from (5.1). Specifically, it was observed there that these charges displayed symmetries corresponding to weight lattice translations for all groups except Sp$(2n)$, n not a power of 2, and moreover, by imposing such symmetries on Sp$(2n)$, one finds an aesthetically pleasing universal formula for the integer y of eq. (5.2), namely

$$y = \mathrm{lcm}\{1, 2, \ldots, \mathrm{h}-1\} ,$$

where h is the Coxeter number of $\mathfrak{g}$. The constraints (5.5) thus provide a theoretical justification for this observation.

Note added. After this note was prepared, another independent twisted K-theory computation appeared [28] which gives explicit results for the (universal covers of the) classical groups and G_2. The torsion parts of these results also appear to agree with the results, eq. (5.2), obtained from the dynamical constraints.

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